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Planck's Constant in the Light of Quantum Logic

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Abstract The goal of quantum logic is the "bottom-top" reconstruction of quantum mechanics. Starting from a weak quantum ontology, a long sequence of arguments leads to quantum logic, to an orthomodular lattice, and to the classical Hilbert spaces. However, this abstract theory does not yet contain Planck's constant $\hbar$. We argue, that $\hbar$ can be obtained, if the empty theory is applied to real entities and extended by concepts that are usually considered as classical notions. Introducing the concepts of localizability and homogeneity we define objects by symmetry groups and systems of imprimitivity. For elementary systems, the irreducible representations of the Galileo group are projective and determined only up to a parameter z, which is given by $z = m/\hbar$, where m is the mass of the particle and $\hbar$ Planck's constant. We show that $\hbar$ has a meaning within quantum mechanics, irrespective of use the of classical concepts in our derivation.

Keywords Planck's constant · Quantum logic · Classical physics

1 Ontological Preliminaries

In 2000, the scientific community of physicists celebrated the 100th anniversary of quantum theory and in particular the birth of Planck's constant, which we usually denote today by $\hbar$. This constant $\hbar$ is widely considered as a characteristic of quantum mechanics and it should appear somewhere in any formulation of this theory. Here, we consider the quantum logic approach to quantum mechanics and find, that in the well known systems of quantum logic the famous constant $\hbar$ does not appear. What is the reason for this apparent deficiency of quantum logic?

The main goal of quantum logic is the "bottom-top" reconstruction of Hilbert lattices, effect algebras, and of quantum mechanics in Hilbert space—and all that without any reference to the actual historical development of the theory [3]. The starting point is a weak

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quantum ontology that describes the most general features of the quantum physical reality. Here we consider three types of ontologies of different strength: The classical ontology $O(C)$, the weaker quantum ontology $O(Q)$, and the unsharp quantum ontology $O(Q^u)$ which is partly stronger than $O(Q)$, since it allows for unsharp joint properties and partly weaker than $O(Q)$, since it does not require value definite properties. In any case, $O(Q^u)$ is weaker than the classical ontology $O(C)$. As to the terminology, we say that an ontology O is stronger than another ontology O', if an entity o contained in O must fulfill more requirements than an entity o' contained in O'.

Many details about these ontologies can be found in the contribution of the present author to the QS-02 meeting [10]. The main result consists in the observation, that the quantum ontologies $O(Q)$ and $O(Q^u)$ can be obtained by convenient relaxations of the classical ontology $O(C)$. The relaxations in question consist in the elimination of metaphysical hypotheses contained in $O(C)$, that are neither justified by rational reasoning nor by experimental evidence. However, the considerations of this paper will show, that it is not sufficient to simply eliminate a certain hypotheses, since it must be replaced by some weaker requirement.

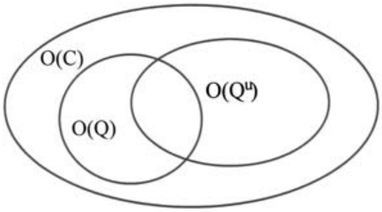
Except from other assumptions, which are not relevant here, the classical ontology is characterized by the following requirements:

– There are individual and distinguishable objects S_i which possess elementary properties P_λ such that either P_λ or the counter property $\bar{P}_\lambda$ pertain to the system. The properties P_λ can always be tested by measurements with the result, that either P_λ or $\bar{P}_\lambda$ pertains to the system.
– Objects S_i are "completely determined", i.e. an object possesses each elementary property P either affirmative (P) or negative $(\bar{P})$. Hence, objects can be individualized by elementary properties and re-identified at later times, if the property of impenetrability is presupposed.

There are important objections against this ontology $O(C)$. The assumptions mentioned are based merely on the metaphysics of the 17th and 18th century and cannot be justified by rational arguments or by experimental evidence. In addition, classical ontology is not in accordance with quantum physics. A quantum system does not possess all elementary properties either affirmative or negative. Instead, only a subset of properties pertains to the system and can simultaneously be determined. These properties are often called "objective" and pertain to the object like in classical ontology. However, quantum objects cannot be individualized and re-identified by their objective properties, since there are not enough such properties. We will not use these empirical results here, but we learn from these considerations that classical ontology has too much structure compared with quantum physics. This observation offers the interesting possibility to formulate the ontology of quantum physics by relaxing some hypothetical assumptions of the classical ontology $O(C)$. We note that no new requirements will be added to the assumptions of $O(C)$. In this sense, our first attempt to a new quantum ontology $O(Q)$ reads:

– If an elementary property P pertains to an object as an objective property, then a test of this property by measurement will lead with certainty to the result P.
– Any elementary property P can be tested at a given object with the result that either P or the counter property $\bar{P}$ pertains to the system.
– Quantum objects are not completely determined. They possess only a few elementary properties either affirmative or negative. Properties, which pertain simultaneously to an object, are called "objective" and "mutually commensurable".

Fig. 1 Interrelations between the ontologies $O(C)$, $O(Q)$ and $O(Q^u)$



The first two requirements are in complete accordance with $O(C)$ whereas the third one is a strong relaxation of the corresponding assumption of $O(C)$. The new quantum ontology $O(Q)$ is not yet in complete accordance with quantum physics for two reasons. First, the most general observables in quantum mechanics—the POV-measures—correspond to unsharp properties that allow for unsharp joint properties, even for complementary observables. Hence, the ontology $O(Q)$ is too restrictive since, generally, it does not allow for joint properties. Second, the requirement of value definiteness of all properties cannot be fulfilled, since the pointer-objectification in the measurement-process cannot be achieved in general. Hence, in this respect the ontology is also not sufficiently restrictive. These two objections against the $O(C)$ can both be taken into account, if $O(Q)$ is replaced by a new quantum ontology $O(Q^u)$ of unsharp properties, provided the degree of unsharpness is conveniently defined. It is a difficult problem to determine exactly, how much unsharpness is necessary for removing the two deficiencies of $O(Q)$ mentioned.

Comparing the three ontologies mentioned, we find that on the one hand $O(Q^u)$ is partly stronger than $O(Q)$, since it allows for unsharp joint complementary properties that are not contained in $O(Q)$. On the other hand, $O(Q^u)$ is partly weaker than $O(Q)$, since value definiteness of properties is not required. However, $O(Q^u)$ is weaker than the classical ontology $O(C)$. These relations are illustrated in Fig. 1.

2 The Quantum Logic Approach

The main goal of the quantum logic approach is the reconstruction of Hilbert lattices and of quantum mechanics in Hilbert space on the basis of the weak quantum ontologies mentioned [3]. Starting from the weak quantum ontology $O(Q)$ we can construct a formal language $\mathcal{S}_Q$ of quantum physics whose syntax leads together with a convenient semantics of truth to the calculus $\mathcal{L}_Q$ of quantum logic. The Lindenbaum–Tarski algebra of $\mathcal{L}_Q$ turns out to be a complete, orthomodular lattice L_Q, which in addition is atomic and fulfills the covering law, if the language is assumed to refer to a single system. We denote this lattice by L_Q^*. Using the Piron–McLaren Theorem [8, 11, 14], and the angle-bisecting condition of Solèr [13], we arrive at the three classical Hilbert spaces and in particular at the complex numbers Hilbert space $\mathcal{H}$(C) of quantum mechanics.

Compared with the classical ontology $O(C)$, a formal classical language $\mathcal{S}_C$ and the classical propositional logic $\mathcal{L}_C$, there are important differences that come from the elimination of the metaphysical hypotheses contained in $O(C)$. In particular, we have sacrificed here the assumption that objects are always "completely determined". As a consequence of this reduction, propositions of the quantum language $\mathcal{S}_Q$ loose their "unrestricted availability" and are in general only restrictedly available. For the calculus $\mathcal{L}_Q$ of quantum logic this relaxation implies the loss of the distributive law. We could go one step further and proceed to the ontology $O(Q^u)$ of unsharp properties by omitting the assumption, that for each property P it is objectively decided, whether P or its counter property $\bar{P}$ pertains to a system.

This relaxation implies that propositions are no longer value definite and that both the "excluded middle" and the "law of contradiction" are no longer formally true [4, 5]. The theory, which we obtain in this way by reducing ontological premises, is an abstract Hilbert space quantum theory of sharp and unsharp properties. It is an empty theory, a formal framework of quantum mechanics, which is presumably universally valid. It is, however, not a priori valid in the strict sense, since the underlying ontologies $O(Q)$ and $O(Q^u)$ do still contain metaphysical premises that are not queried here. The abstract quantum theory, which is reconstructed on the basis of the weak quantum ontology $O(Q^u)$ is, however, closer to the truth than the theory based on $O(Q)$ and in any case closer to the truth than the classical mechanics based on the classical ontology $O(C)$.

3 In Search of Planck's Constant

Within the quantum logic approach quantum mechanics in Hilbert space appears as an abstract and empty theory which is based on the weak quantum ontology and thus presumably universally valid. Hence, we expect first of all to discover somewhere in this theory Planck's constant $\hbar$, which is widely considered as a characteristic of quantum mechanics and as a number, that indicates the border line between the quantum world and the classical world. However, within the quantum-logic approach there is no classical world [10] and hence no border line between the two worlds, from which we could read off Planck's constant. Hence, there is no hope to find the constant $\hbar$ within the domain of abstract quantum theory in Hilbert space. In order to discover Planck's constant in the realm of quantum logic, we must extend the abstract and empty theory by incorporating real entities into the theory. We will find that "objects" or "particles" can be comprehended if the abstract theory is extended by concepts that are usually considered as classical notions. Since intuitively, particles are objects that are somehow localized in space, we consider first the concepts of *localizability* and *homogeneity*.

3.1 Localizability

Let Δ be a domain of the physical space R. If $R = R^{(1)}$ is one dimensional, the domains Δ considered are Borel sets of the real line, i.e. $\Delta \in \mathcal{B}(R)$. Let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on a Hilbert space $\mathcal{H}$. The mapping

$$\mathrm{E} : \mathcal{B}(R) \to \mathcal{L}(\mathcal{H});\ \ \Delta \mapsto \mathrm{E}\{\Delta\}$$

is a projection valued measure (PV-measure), if $\mathrm{E}\{\Delta\} = \mathrm{E}\{\Delta\}^* = \mathrm{E}\{\Delta\}^2$ for all $\Delta \in \mathcal{B}(R)$, $\mathrm{E}\{R\} = \mathrm{I}$, and $\mathrm{E}\{\cup\Delta_i\} = \sum \mathrm{E}\{\Delta_i\}$. According to the spectral theorem, this PV-measure leads to a self-adjoint operator, the position operator Q. More generally, we could start with a non-empty set Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω, and hence on a measurable space $(\Omega, \mathcal{F})$. A normalized positive operator valued measure (POV-measure) can then be defined by

$$\mathrm{E} : \mathcal{F} \to \mathcal{L}(\mathcal{H}) \quad \text{on } (\Omega, \mathcal{F}),$$

where $\mathrm{E}\{\mathrm{X}\} \geq 0$, $X \in \mathcal{F}$, $\mathrm{E}\{\Omega\} = \mathrm{I}$ and $\mathrm{E}\{\cup \mathrm{X}_i\} = \Sigma \mathrm{E}\{\mathrm{X}_i\}$ for disjoint sequences $(X_i) \in \mathcal{F}$.

3.2 Homogeneity

Homogeneity and isotropy are features of the physical space, which show that the physical space has no observable properties. In a one dimensional space this means, that a translation by an amount α

$$g_\alpha : \Delta \mapsto g_\alpha(\Delta) = \{\lambda : (\lambda - \alpha) \in \Delta\}$$

with $\Delta \in \mathcal{B}(R)$, is a symmetry transformation. If the physical space is homogeneous then there exists a unitary operator U_α, depending on α, $U_\alpha : E \mapsto U_\alpha^{-1} E U_\alpha$ such that

$$E\{g_\alpha(\Delta)\} = U_\alpha^{-1} E\{\Delta\} U_\alpha$$

where $E\{\Delta\}$ is the projection operator mentioned above. We can choose the parameter α such that it is additive. i.e. $U_\alpha U_\beta = U_{\alpha+\beta}$. According to Stone's theorem and under this condition a self-adjoint operator P with $U_\alpha = \exp(i\alpha P)$, the displacement operator, is uniquely determined.

3.3 Canonical commutation relations

If we consider the position operator Q as generator of a one-parameter group with parameter β, we get $V_\beta = \exp(i\beta Q)$. Together with the corresponding expression $U_\alpha = \exp(i\alpha P)$ for the displacement operator P, we find the canonical commutation relations in the Weyl formulation

$$U_\alpha V_\beta = e^{i\alpha\beta} V_\beta U_\alpha.$$

For a dense subset D of the entire Hilbert space the Weyl commutation relations imply that the operators P and Q satisfy the relation

$$[Q, P] f = i f \quad \text{for all } f \in D.$$

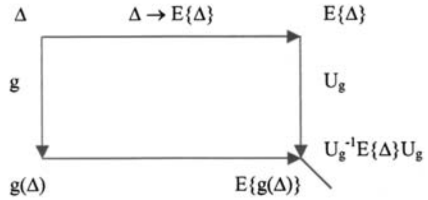
3.4 Physical objects

The notion of an object is intimately related with the equivalence of active and passive space-time transformations and with the covariance of observables under these transformations. Indeed, if we understand by an object an entity of the external reality that exists objectively and independent of the observing subject and his measurement devices, then it should not matter whether the object is (actively) transformed by a translation in space, say—or whether the apparatus and its coordinates are (passively) transformed in the opposite direction. If we combine this idea with the concepts of localization and homogeneity mentioned, we can characterize an "object" in the following way:

Let M be a topological space, the configuration space of the intended object and G a locally compact transformation group that acts transitively on M. Here, we think preferably of the Galileo group G and its one-parameter subgroups of space translations and velocity boosts. An element $g \in G$ induces a one-to-one and continuous mapping of M to itself

$$g: \Delta \to g(\Delta),\ M \mapsto M,$$

Fig. 2 Covariance diagram



with $\Delta \in \mathcal{B}(M)$. A projection valued measure E: $\Delta \to E\{\Delta\}$ leads to a unitary representation of the group, $g \to U_g$ with

$$U_g\colon E \mapsto U_g^{-1} E U_g.$$

We can now express the requirement of objectivity (of the intended object) by the covariance diagram in Fig. 2.

For an element of the external reality it should not matter whether we transform first the domain Δ by a mapping g of M to itself (passive transformation) and go in a second step (corresponding to a PV-measure) from $g(\Delta)$ to $E\{g(\Delta)\}$;—or whether we go first from Δ to the projection operator $E\{\Delta\}$ and in a second step (corresponding to a unitary representation U_g) from $E\,\{\Delta\}$ to $U_g^{-1}E\{\Delta\}U_g$ (active transformation).

This requirement, which expresses the equivalence of active and passive transformations, means that the covariance diagram in Fig. 2 commutes, i.e.

$$E\{g(\Delta)\} = U_g^{-1} E\{\Delta\} U_g.$$

Observables E that fulfill this "covariance postulate" correspond to properties of the object in question that transform covariant under the transformations of the Galileo group. In other words, a quantum object is carrier of properties $E\{\Delta\}$ which transform covariant under the Galileo transformations [9, 12].

3.5 Elementary particles

On the basis of this general concept of a quantum object as carrier of properties $E\{\Delta\}$ of the orthomodular lattice L_Q, we can specify this concept by considering different classes. Different representations $g \to U_g$ of elements $g \in G$ of the 10-parameter Galileo group by automorphism U_g (on the lattice L_Q of projection operators) correspond to different kinds of objects. In particular, the elementary objects are given by irreducible representations $g \to U_g$. However, there are no irreducible unitary *true* representations $g \to U_g$ of the Galileo group but only projective ones that contain a real, yet undetermined parameter z.[1]

For further illustrating this result, we mention briefly a few technical steps [7]. From the position operator Q a self-adjoint operator $\dot{Q}$ for the velocity can be obtained by formal differentiation with respect to the time t, i.e. by $\dot{Q} = i[H, Q]$, where H is the evolution

[1] It should be emphasized, that this way of reasoning for the constitution of objects is not restricted to PV-measures, i.e. to sharp observables, since it can easily be extended and generalized to unsharp observables in the sense of POV-measures. Unsharp properties are then given by "effects" in Hilbert space, the algebra of which will be denoted here by $E(\mathcal{H})$. [2, p. 25], [5]. Hence, in the covariance diagram (Fig. 2), the PV-measures must be replaced by POV-measures and the lattice of projection operators by the effect algebra $E(\mathcal{H})$. In this case, objects are carriers of the most general observables, given by POV-measures [9, p. 1623], [2, p. 52].

operator which is not yet fully determined at this point. For motivating the next step, we consider again the classical situation. If we change the reference system K to a new system K' which moves with the constant velocity $v = v(K, K')$ relatively to K, the velocity $\dot{Q}$ and the position operator Q will change according to the transformation $\dot{Q} \mapsto \dot{Q} + v$, $Q \mapsto Q$. If also in quantum physics this velocity boost transformation is considered as a symmetry transformation, then there exists a one-parameter unitary group G_v, such that

$$\dot{Q} + v = \mathrm{G}_v \dot{Q} \mathrm{G}_v^{-1} \quad \text{and} \quad \mathrm{G}_v \mathrm{G}_{v'} = \mathrm{G}_{v+v'}.$$

Since the system is elementary and G_v commutes with Q, we may write $\mathrm{G}_v = \exp(\mathrm{i}vf(\mathrm{Q}))$ where f is a Borel function on the real line. For combining the velocity boosts with the displacements mentioned above we define a two-parameter family of unitary operators $\mathrm{T}(\alpha, v)$ such that

$$Q + \alpha = \mathrm{T}(\alpha, v) Q \mathrm{T}^{-1}(\alpha, v),$$
$$\dot{Q} + v = \mathrm{T}(\alpha, v) \dot{Q} \mathrm{T}^{-1}(\alpha, v).$$

Hence, $\mathrm{T}(\alpha, v)$ is a projective representation of the two-dimensional translation group whose arbitrary phase factor can be written in the form

$$\mathrm{T}(\alpha_1, v_1)\mathrm{T}(\alpha_2, v_2) = e^{i\frac{z}{2}(\alpha_1 v_2 - \alpha_2 v_1)}\mathrm{T}(\alpha_1 + \alpha_2, v_1 + v_2)$$

where $z \neq 0$ is an arbitrary real constant which distinguishes different inequivalent projective representations. We can re-identify the one-parameter subgroups U_α und G_v by the relations $\mathrm{U}_\alpha = \mathrm{T}(\alpha, 0)$ and $\mathrm{G}_v^{-1} = \mathrm{T}(0, v)$, and by means of the commutation relations we find U_α $\mathrm{G}_v^{-1} = \exp(\mathrm{i}z\alpha v)$ G_v^{-1} U_a and $\mathrm{G}_v^{-1} = \exp(\mathrm{i}z\alpha Q)$. From this relation we obtain in a few steps [7] $\dot{Q} = P/z$ and $P/z = \mathrm{i}[H, Q]$ where H is the most general evolution parameter which is compatible with the principle of Galileo invariance.

In order to identify the parameter z, we refer to some classical aspects of an elementary object. If the object considered is localizable, we will call it an elementary particle. The parameter z is often interpreted as the inertial mass m of the particle in question. This is, however, not quite correct. A more detailed investigation that refers to the classical motion and to the dynamics of the particle shows, that $z = m/\hbar$, where m is the inertial mass and $\hbar$ is a universal constant. Indeed, if we describe the classical motion by the movement of a point in the configuration space (given by the real line), then we can identify the classical motion with the motion of the expectation value x of the position operator Q.

Let W be a state operator with $W \in \mathcal{T}(\mathcal{H})_1^+$, where $\mathcal{T}(\mathcal{H})_1^+$ is the set of *positive trace one operators*. Then we have $x = \mathrm{tr}\{WQ\}$, and for the velocity we find $dx/dt = \mathrm{tr}\{W\dot{Q}\} = i \cdot \mathrm{tr}\{W[H, Q]\} = \mathrm{tr}\{WP\}/z$. Here we made use of the relations $\dot{Q} = P/z = \mathrm{i}[H, Q]$ mentioned above. For the momentum we obtain

$$m \cdot dx/dt = \mathrm{tr}\{WP\}m/z = \hbar \cdot \mathrm{tr}\{WP\}.$$

At this preliminary stage of the discussion the constant $\hbar$ can be identified as connecting the displacement operator P with the momentum operator p of the particle such that $p = \hbar P$ and can be determined experimentally (at least in principle) as

$$\hbar = m/z = p/P = 1.05 \times 10^{-27}\ \mathrm{erg\,s}.$$

Summarizing these arguments we find, that the decisive steps in our search for Planck's constant $\hbar$ made use explicitly of classical concepts:

1. The concepts of localizability and homogeneity of the physical space
2. The definition of objects making use of covariance diagrams that distinguish explicitly representations of the Galileo group by transformations of the physical space-time and by automorphism U_g on the algebra of projection operators.
3. The classical concept of movement of a point in the configuration space.

Hence, our first, still preliminary result is, that for discovering the physical meaning of Planck's constant $\hbar$, in addition to the abstract quantum logic, classical concepts must be taken into account. However, this result would invalidate the idea of an autonomous quantum world without any recourse to a classical world.

4 The Meaning of $\hbar$ in the Quantum World

Within the framework of the quantum theory in Hilbert space, we can derive a relation that is of particular importance for the constant $\hbar$. We think of the uncertainty relation, in particular if it is formulated in terms of unsharp observables, i.e. of POV-measures, [1], [2, p. 59 f. and 107 f.], [6]. On the basis of the requirement of Galileo covariance for position q and momentum p the uncertainty relation

$$\delta q \cdot \delta p \geq \hbar/2$$

can be derived, where the meaning of the expressions δq and δp differs in various interpretations of quantum mechanics. Here we are interested in the "Heisenberg interpretation" of the uncertainty relations, i.e. an individualistic interpretation of these relations in terms of unsharp observables q and p [2, p. 108]. The number $\hbar$ may then be considered as the *smallest possible degree of inaccuracy of jointly measured observables q and p that are probabilistically complementary.* Obviously, this meaning of $\hbar$ can be expressed exclusively in terms of quantum physics and without any recourse to the classical world.

As to quantum-logic, the meaning of $\hbar$ must be expressed in terms of the abstract language $\mathcal{S}_Q$ and of the formal logic $\mathcal{L}_Q$. Compared with the language $\mathcal{S}_C$ of classical physics, the main restriction of quantum language is the restricted availability of propositions in a formal proof process. Proof processes are formulated either by a derivation within the framework of a calculus or by a dialog according to the rules of the material or formal dialog game. If after a material proof of a proposition A, another proposition B was successfully shown to be true, the previously proved proposition A is no longer available except proposition A and B are commensurable. For unsharp propositions, these strict alternatives can be considerably be relaxed, since even probabilistically complementary, unsharp propositions are not strictly incommensurable. Consequently, the degree to which the proposition A is still available after a proof of B, depends on the degree of commensurability of A and B. Hence, in the spirit of the uncertainty relation for individual unsharp propositions, the constant $\hbar$ can be identified here as a measure for *the smallest possible unavailability of the unsharp complementary propositions A and B*. In other words, Planck's constant determines the smallest possible unavailability, and in this sense $\hbar$ is a universal constant in the realm of quantum-logic.

We can go one step further to the ontology $O(Q^u)$ of unsharp properties. Compared with the classical ontology $O(C)$, the main restriction of the quantum ontology $O(Q)$ is, that objects are not completely determined. This restriction is, however, too strong since for the most general observables we must allow for unsharp joint properties even for probabilistically complementary observables. This argument is taken account of in the ontology $O(Q^u)$

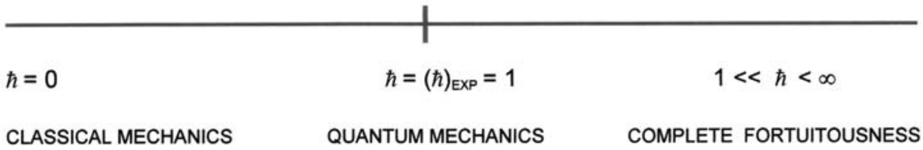


Fig. 3 Planck's constant shows the position of quantum mechanics

of unsharp properties. Objects of $O(Q^u)$ are "partially determined" in the sense that all possible predicates pertain at least unsharp to the system. This is, however, only a qualitative characterization of the ontology $O(Q^u)$. Quantitatively, and in the spirit of the uncertainty relation we can say, that the *minimal degree of unsharpness of probabilistically complementary properties which pertain jointly to a system*, is given by $\hbar$. This is the meaning of Planck's constant on the level of quantum ontology.

Hence, we find that Planck's constant $\hbar$ is also an intrinsic characteristic of quantum ontology and thus of the quantum world at all, that describes "*the largest possible degree of joint determination of unsharp complementary properties*". In contrast to classical systems, quantum systems are only partially determined, where the largest possible degree of partial determination is measured by $\hbar$. Since in this way we can identify $\hbar$ as an intrinsic feature of quantum physics and express this feature exclusively in terms of quantum physics, we could forget about the long detour on our way to Planck's constant making use of several classical concepts. In other words, we can through away "Wittgenstein's ladder" whose steps consist in the present case of various classical concepts. On the basis of these results, we can now try to answer the question posed in the introduction: What is the reason, why in the operational approach to quantum logic the constant $\hbar$ does not appear in the first instance? For a bottom-top reconstruction of quantum mechanics the reduced quantum ontologies $O(Q)$ and $O(Q^u)$ are too general and not sufficiently specific for a complete reconstruction of quantum logic and quantum mechanics. These theoretical structures can be obtained only up to an unknown real parameter, whose numerical value must be determined empirically and turns out to be $\hbar$. Hence, on the long scale between classical physics and complete fortuitousness, Planck's constant determines the actual position of quantum logic and quantum physics (Fig. 3).

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